Mid-Term Exam | ISI Bangalore | September 2014

Algebra III | 100 Points

Notes.

(a) Begin each answer on a separate sheet and ensure that the answers to all the parts to a question are arranged contiguously.

(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers $\mathbb{C}$ = complex numbers.

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1. [10 points] Let $f\colon \mathbb{C}[X,Y] \to \mathbb{C}[t]$ be the map between polynomial rings over $\mathbb{C}$ such that f is identity on $\mathbb{C}$ and $f(X) = t^2$, $f(Y) = t^5$. Prove that the kernel of f is generated by $Y^2 - X^5$.

2. [15 points]
   (i) Let R be a ring. Prove that for ideals I, J in R, if $I + J = R$, then $I \cdot J = I \cap J$.
   (ii) In the ring $R = \mathbb{Z}[\sqrt{-5}]$ prove that $(3, 1-\sqrt{-5}) \cap (2, 1+\sqrt{-5}) = (1-\sqrt{-5})$.

3. [30 points] Let $\mathbb{F}_3$ denote the field of 3 elements. In each of the following cases of $f \in \mathbb{F}_3[X]$, determine whether the ring $R = \mathbb{F}_3[X]/(f)$ is a field, whether it has a nonzero nilpotent, and whether it has a nontrivial idempotent.

   (a) $f(X) = X^2 + X + 1$ (b) $f(X) = X^2 + 1$ (c) $f(X) = X^2 + 2$

4. [15 points]
   Find two maximal ideals containing $I = (X^2 + Y^2 - 1, X + Y - 1)$ in $R = \mathbb{C}[X,Y]$. Prove that these two are the only ideals containing I apart from I and R.

5. [7 points] Find all the units in $\mathbb{Z}[\sqrt{-3}]$.

6. [15 points] Prove that $\mathbb{Z}[\omega]$ is a Euclidean domain.

7. [8 points] Give an example of a ring R that is not a domain and in which every ideal is principal.